# Nonclassical Linearization Criteria in Nonlinear Stochastic Dynamics

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*In classical stochastic linearization method, the nonlinear force is replaced by an equivalent linear one. Several nonclassical schemes were suggested in recent years, based on potential or complementary energy criteria. Here, these criteria are compared with each other, and the classical stochastic linearization scheme, to determine their efficacy.* [DOI: 10.1115/1.4000899]

## 1 Introduction

Stochastic nonlinear dynamics has reached the level of considerable maturity, in the recent decade or so. There have been new exact solutions obtained, whereas for other problems—and they naturally are the overwhelming majority—powerful approximate methods have been developed. The stochastic linearization technique occupies the central role among the approximate methods. It was suggested almost simultaneously by Booton [1], Kazakov [2] and Caughey [3,4] over half a century ago. Perhaps in over 99% of the studies on stochastic linearization, the requirement of the minimum mean-square difference between the nonlinear restoring force and its linear counterparts in the replacing system is utilized. Kazakov [2] also proposed a criterion that demands the equality of the mean-square values of the nonlinear force and its linear counterpart.

For extensive reviews of this method the reader can consult with papers by Falsone and Ricciardi [5], Socha [6], and Bernard [7]. Elishakoff [8] gave a reinterpretation of the classical method. For the purpose of this study it is pertinent to mention the alternative criterion, based on energy, as suggested by Wang and Zhang [9], Zhang et al. [10], Elishakoff et al. [11,12], and Falsone and Elishakoff [13]. This method was realized for various applications by Fang et al. [14], Muravyov et al. [15], Zhang [16], Zhang et al. [17], and Zhang and Zhang [18]. Elishakoff and Bert [19] developed a complementary energy criterion, but no numerical evaluation was performed to compare it with the potential energy-based procedures.

In this paper we provide a systematic comparison of the energy criteria on the example of the Duffing oscillator. This nonlinear oscillator is chosen since there is an exact solution readily available for it. This allows a straightforward comparison to be conducted of the approximate solutions with the exact one, allowing to discern the relative efficacy of the approximations.

## 2 Analysis

To gain more insight into the appraisal of various energy-based techniques, let us consider a specific nonlinear system, namely, the oscillator with polynomial nonlinearity

$$m\ddot{X} + c\dot{X} + k_0 X + k_n X^{2n+1} = f(t) \tag{1}$$

where $X(t)$ is the displacement, $m$ is the mass, $c$ is the damping, $k_0$ is the linear stiffness coefficient, $n$ is taken as a positive integer, for the sake of the simplicity, and $k_n$ is the nonlinear stiffness coefficient associated with the nonlinear behavior of the system. When $n=1$, we get a particular case of the Duffing oscillator. For general nonwhite noise excitation, there is no exact solution available for the probability density of $X(t)$, or even for the mean-square displacement $E(X^2)$, where $E(\ )$ denotes the mathematical expectation. The potential energy of the system reads

$$P(X) = \frac{1}{2} X F(X) \tag{2}$$

where $F(X)$ is the restoring force

$$F(X) = k_0 X + k_n X^{2n+1} \tag{3}$$

Thus

$$P(X) = \frac{1}{2} k_0 X^2 + \frac{1}{2(n+1)} k_n X^{2(n+1)} \tag{4}$$

The complementary energy $C(X,t)$ defined as

$$C(X) = XF(X) - P(X) \tag{5}$$

is obtained as follows:

$$C(X) = X(k_0 X + k_n X^{2n+1}) - P(X) = X(k_0 X + k_n X^{2n+1}) - \left(\frac{1}{2}k_0 X^2 + \frac{1}{2(n+1)} k_n X^{2(n+1)}\right) = \frac{1}{2} k_0 X^2 + \left(\frac{2n+1}{2(n+1)}\right) k_n X^{2n+1} \tag{6}$$

We replace the original system (1) by its linear counterpart

$$m\ddot{X} + c\dot{X} + k_{\mathrm{eq}} X = f(t) \tag{7}$$

with the attendant equivalent potential energy designated as $P_{\mathrm{eq}}(X)$

$$P_{\mathrm{eq}}(X) = \frac{1}{2} k_{\mathrm{eq}} X^2 \tag{8}$$

The associated equivalent complementary energy

$$C_{\mathrm{eq}}(X) = XF_{\mathrm{eq}}(X) - P_{\mathrm{eq}}(X) \tag{9}$$

naturally coincides with the potential energy of the replacing system $P_{\mathrm{eq}}$.

Let us now introduce the orthogonality based stochastic linearizations [8].

## 3 Potential Energy Linearization

We demand first that the difference between potential energies in the original and replacing systems be orthogonal to $X^{2n}$, i.e.

$$E\{[P(X) - P_{\mathrm{eq}}(X)]X^{2n}\} = 0 \tag{10}$$

or

$$E\left\{\left[\frac{1}{2}k_0 X^2 + \frac{1}{2(n+1)} k_n X^{2(n+1)} - \frac{1}{2} k_{\mathrm{eq}} X^2\right] X^{2n}\right\} = 0 \tag{11}$$

leading to the following expression of the equivalent stiffness

$$k_{\mathrm{eq}} = k_0 + \frac{1}{n+1} k_n \frac{E(X^{2(2n+1)})}{E(X^{4n})} \tag{12}$$

This formula, although derived differently, coincides with the expression proposed by Wang and Zhang [9].

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## 4 Complementary Energy Linearization

In this context, we require that the difference between the complementary energies in the original and replacing systems be orthogonal to $X^{2n}$, i.e.

$$E\{[C(X) - C_{eq}(X)]X^{2n}\} = 0 \tag{13}$$

$$E\left\{\left[\frac{1}{2}k_0X^2 + \frac{2n+1}{2(n+1)}k_nX^{2(n+1)} - \frac{1}{2}k_{eq}X^2\right]X^{2n}\right\} = 0 \tag{14}$$

This leads to the following equivalent stiffness:

$$k_{eq} = k_0 + \frac{2n+1}{n+1}k_n\frac{E(X^{2(2n+1)})}{E(X^{4n})} \tag{15}$$

The particular case for $n=1$ of this formula was derived by Elishakoff and Bert [19] but no numerical evaluation of its efficacy was studied. We will compare the efficacy of the approximations obtained with Eqs. (10) and (13), with the exact solution available in the case that $f(t)$ is a zero-mean Gaussian process with white noise autocorrelation function.

## 5 Comparison of Two Energy Criteria

Due to the assumption of normal distribution for $X(t)$ we have

$$E(X^{2n}) = (2n-1)!![E(X^2)]^n \tag{16}$$

Hence, Eqs. (12) and (15) on be cast in the same form

$$k_{eq} = k_0 + \alpha k_n E(X^2) \tag{17}$$

where

$$\alpha = \begin{cases} \dfrac{4n+1}{n+1}, & \text{for potential energy linearization} \\ (4n+1)\dfrac{2n+1}{n+1}, & \text{for complementary energy linearization} \end{cases} \tag{18}$$

The mean-square response of the replacing system reads

$$E(X^2) = \frac{\pi S}{ck_{eq}} \tag{19}$$

Substituting Eq. (17) into Eq. (19) we arrive at

$$E(X^2) = \frac{\pi S}{ck_0}\frac{1}{1 + \alpha(k_n/k_0)E(X^2)} \tag{20}$$

which leads to the quadratic equation

$$\alpha(k_n/k_0)[E(X^2)]^2 + E(X^2) - E(X_0^2) = 0 \tag{21}$$

where

$$E(X_0^2) = \frac{\pi S}{ck_0} \tag{22}$$

is the mean-square response of the system without the nonlinear spring altogether. Equation (21) has a single positive root for $E(X^2)$, reading

$$E(X^2) = \frac{\sqrt{1 + 4\alpha(k_n/k_0)E(X_0^2)} - 1}{2\alpha(k_n/k_0)} \tag{23}$$

If the nonlinearity is small, i.e., if

$$4\alpha(k_n/k_0)E(X_0^2) \ll 1 \tag{24}$$

then the asymptotic solution $E(X_0^2)$ is obtained, coinciding with the linear solution when $\sqrt{1+4\alpha(k_n/k_0)E(X_0^2)}$ is approximated as $1+2\alpha(k_n/k_0)E(X_0^2)$. When the value $4\alpha(k_n/k_0)E(X_0^2)$ is moderate, we approximate as follows:

$$\sqrt{1 + 4\alpha(k_n/k_0)E(X_0^2)} \approx 1 + 2\alpha(k_n/k_0)E(X_0^2) - \frac{1}{2}\alpha^2(k_n/k_0)^2E^2(X_0^2) \tag{25}$$

resulting in

$$E(X^2) \approx E(X_0^2) - \frac{1}{4}\alpha(k_n/k_0)E^2(X_0^2) \tag{26}$$

## 6 Mean-Square Equality Criteria

Alternative criteria may be developed. Elishakoff et al. [20] suggested that the mean-square energies in the original and replacing systems equal

$$E[P^2(X)] = E[P_{eq}^2(X)] \tag{27}$$

This requirement results in the following value of $k_{eq}$ for the polynomial oscillator at hand

$$k_{eq} = \left[k_0^2 + \frac{1}{(n+1)^2}k_n^2\frac{E(X^{4(n+1)})}{E(X^4)} + \frac{2}{(n+1)}k_0k_n\frac{E(X^{2(n+2)})}{E(X^4)}\right]^{1/2} \tag{28}$$

We can demand that the mean-square complementary potential energies be equal, i.e.

$$E[C^2(X)] = E[C_{eq}^2(X)] \tag{29}$$

For the polynomial oscillator this requirement is associated with the following equivalent stiffness:

$$k_{eq} = \left[k_0^2 + \left(\frac{2n+1}{(n+1)}\right)^2k_n^2\frac{E(X^{4(n+1)})}{E(X^4)} + 2\left(\frac{(2n+1)}{(n+1)}\right)k_0k_n\frac{E(X^{2(n+2)})}{E(X^4)}\right]^{1/2} \tag{30}$$

With Eq. (16) taken into account, we can rewrite Eqs. (26) and (28) in the similar manner

$$k_{eq} = k_0\left[1 + \beta\left(\frac{k_n}{k_0}\right)^2[E(X^2)]^{2n} + \gamma\frac{k_n}{k_0}E(X^2)^n\right]^{1/2} \tag{31}$$

where

$$\beta = \begin{cases} \dfrac{1}{(n+1)^2}\dfrac{(4n+3)!!}{3}, & \text{for potential energy linearization} \\ \left(\dfrac{2n+1}{(n+1)}\right)^2\dfrac{(4n+3)!!}{3}, & \text{for complementary energy linearization} \end{cases} \tag{32}$$

$$\gamma = \begin{cases} \dfrac{2}{(n+1)}\dfrac{(2n+1)!!}{3}, & \text{for potential energy linearization} \\ 2\left(\dfrac{(2n+1)}{(n+1)}\right)\dfrac{(2n+1)!!}{3}, & \text{for complementary energy linearization} \end{cases} \tag{33}$$

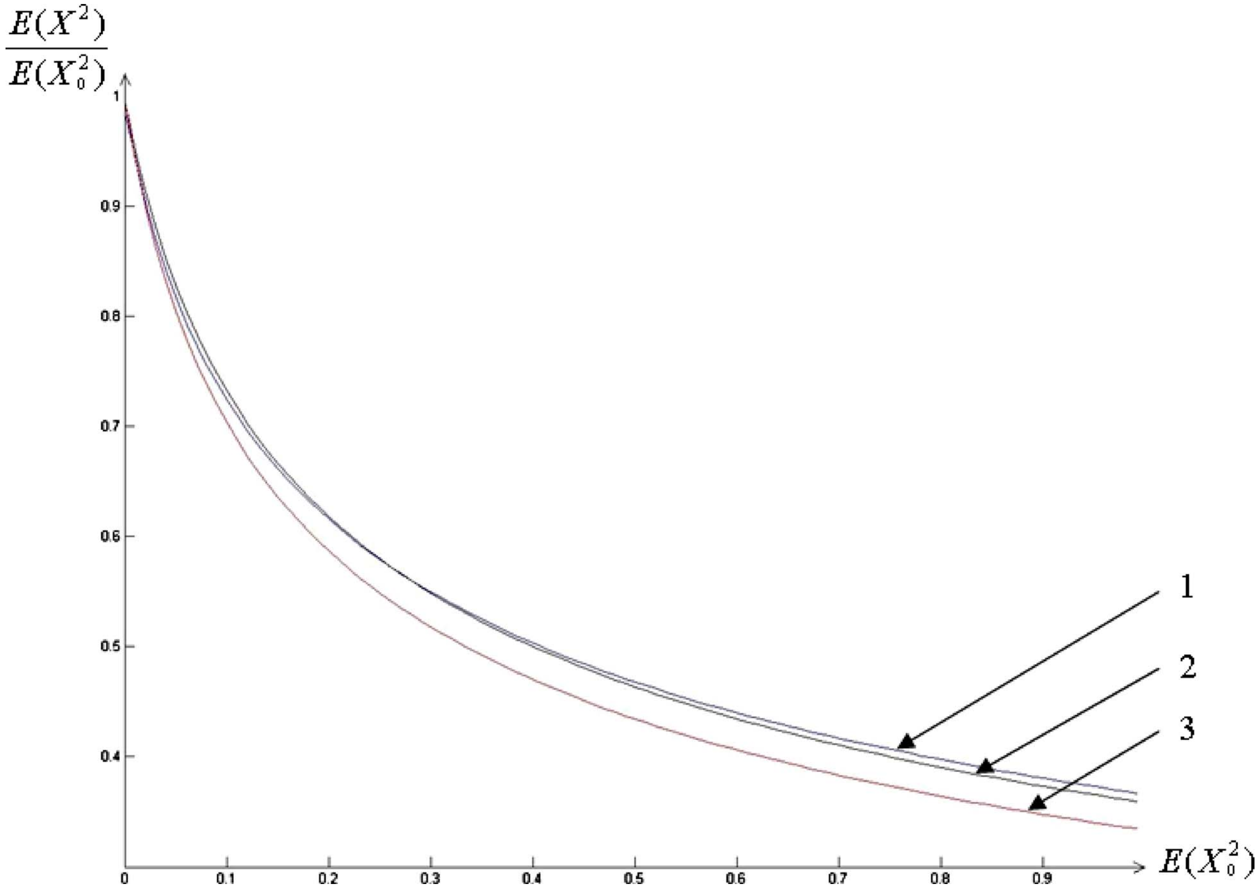


**Fig. 1 Comparison of the approximate solutions with the exact results: (1) exact solution; (2) potential energy criterion in Eq. (10); and (3) classical stochastic linearization results**

## 7 Particular Case: Duffing Oscillator

For the case where $n=1$, we obtain the Duffing oscillator. The coefficients for the previous linearization method read

$$\beta=\begin{cases}\dfrac{35}{4}, & \text{for potential energy linearization}\\ \dfrac{315}{4}, & \text{for complementary energy linearization}\end{cases} \tag{34}$$

$$\gamma=\begin{cases}1, & \text{for potential energy linearization}\\ 3, & \text{for complementary energy linearization}\end{cases} \tag{35}$$

Density of exact solution is given by

$$p_{X(t)}(u) = A \exp\left(\frac{-P(u)}{\pi S_0}\right) \tag{36}$$

where $P(u)=k_0u^2/2+k_1u^4/4$ is the potential energy of the system, and $A$ is a normalization constant evaluated from

$$A^{-1} = \int_{-\infty}^{+\infty} \exp\left(\frac{-P(u)}{\pi S_0}\right) du \tag{37}$$

Thus Eq. (36) becomes

$$p_{X(t)}(u) = \frac{\exp\left(\dfrac{-1}{\pi S_0}[k_0u^2/2 + k_1u^4/4]\right)}{\displaystyle\int_{-\infty}^{+\infty} \exp\left(\dfrac{-1}{\pi S_0}[k_0v^2/2 + k_1v^4/4]\right) dv} \tag{38}$$

It is convenient to introduce a normalizing constant $\sigma_0^2=\pi S_0/k_0$, which is the variance of $X(t)$ for the limiting linear case $k_1=0$. Using a change in the variable of $u=w\sigma_0$ then gives the probability density function as

$$p_{X(t)}(u) = \frac{\exp(-w^2/2 - \alpha w^4/4)}{\int_{-\infty}^{+\infty}\exp(-v^2/2 - \alpha v^4/4)dv} \tag{39}$$

with $\alpha=k_n\sigma_0/k_0$.

We introduce notations

$$k_nu^4/4\pi S_0 = t^4, \quad y = k_0/4\sqrt{\pi S_0 k_n} \tag{40}$$

Equation (37) is rewritten as

$$A^{-1} = Z_1(y) = 2\int_0^{+\infty} e^{-t^4-4yt^2} dt \tag{41}$$

where the function $Z_1(y)$ was introduced by Stratonovich [21].

The mean-square response reads

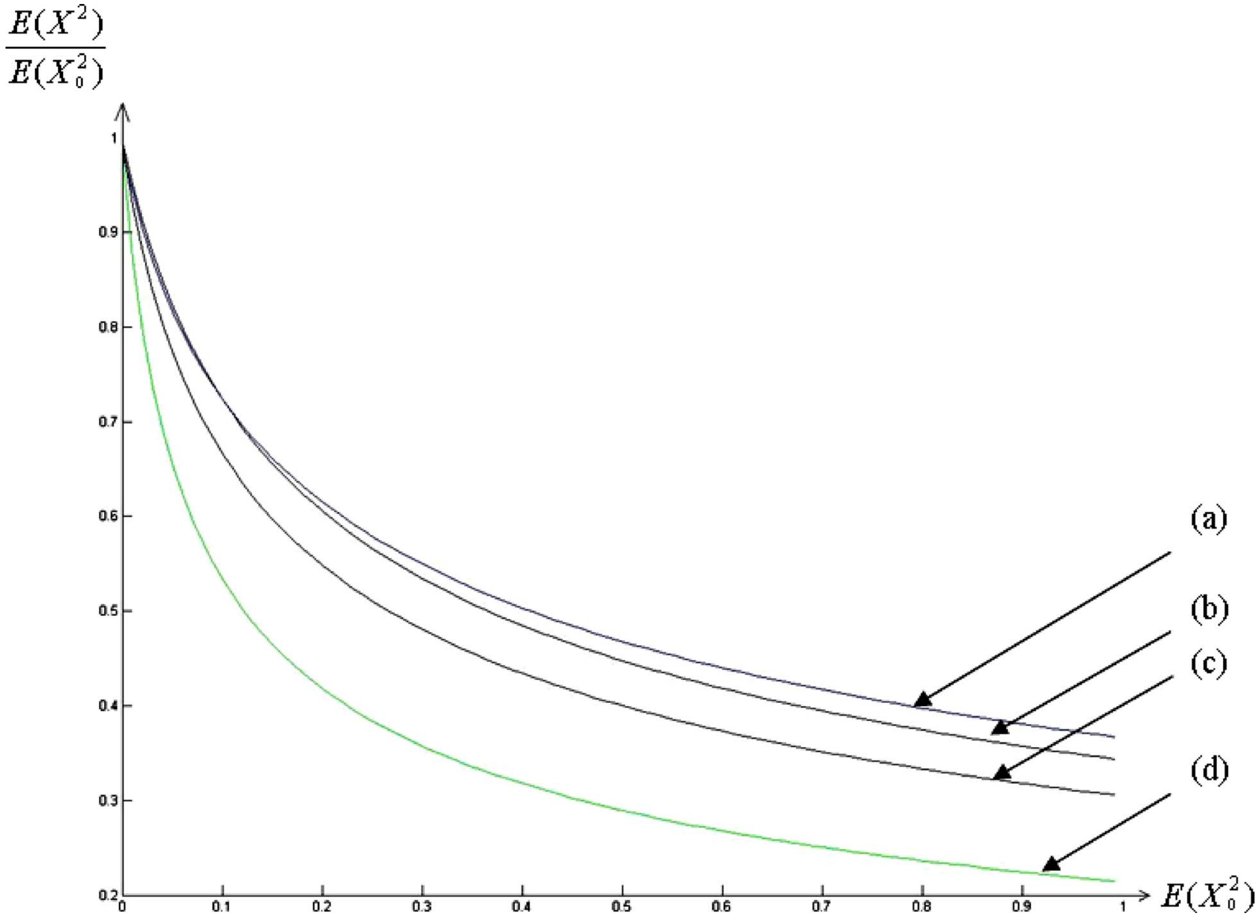


**Fig. 2 Contrast of the potential and complementary energy criteria with the exact solution: a—exact solution; b—criterion based on equality of mean-squares of potential energies in Eq. (27); c—criterion based on complementary energy in Eq. (13); and d—criterion based on equality of mean-squares of complementary energies**

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 p_{X(t)}(u)du = A^{-1}\int_{-\infty}^{+\infty} u^2 \exp(-P(u)/\pi S_0)du \tag{42}$$

with notations in Eq. (40), the mean-square response becomes

$$E(X^2) = 2\sqrt{\pi S_0/k_n}A^{-1}Z_2(y) \tag{43}$$

where

$$Z_2(y) = 2\int_0^{+\infty} e^{-t^4-4yt^2}t^2dt \tag{44}$$

is a second Stratonovich [21] function.

For $y>0$, the functions $Z_1(y)$ and $Z_2(y)$ turn out to be connected with the modified Bessel functions of fractional order. The functions $Z_1(y)$ and $Z_2(y)$ were numerically evaluated by Stratonovich [21, p. 545]. Here we reproduce his results along with present derivations

$$Z_1(y) = \sqrt{y}e^{2y^2}K_{1/4}(2y^2)$$
$$Z_2(y) = y\sqrt{y}e^{2y^2}[K_{3/4}(2y^2) - K_{1/4}(2y^2)] \tag{45}$$

Analogous expression for $A^{-1}$ was found by Piszczek and Niziol [22, p. 173]. For the mean-square response some other authors use the parabolic cylinder function $D_{-3/2}$.

To shed more light on the accuracy of the nonclassical approximations, it is necessary to conduct comparison between the exact solution and approximate ones. Figure 1 shows a comparison between the exact solution and various energy criteria.

Figure 1 represents the comparison of the exact solution with the potential energy linearization as performed in Eq. (10). Curve 1 is en exact solution. Curve 2 is based on Eq. (10), whereas curve 3 is based on classical linearization. Energy criterion is observed as superior to the force linearization.

Figure 2 contrasts various energy criteria. Curve (a) depicts the exact solution, curve (b) is associated with the criterion of equal mean-square potential energies in Eq. (27), and curve (c) depicts the results based on the orthogonality of the mean-square difference of complementary energies, whereas curve (d) is based on the equality of the mean-squares of complementary energies. As seen, the criterion based on the equality of the mean-square potential energies is closer to the exact solution than that based on the equality of the mean-square complementary energies. Likewise, comparison of Fig. 1, curve 2 and Fig. 2, curve (b), suggests that the potential energy orthogonalization criterion (Eq. (10)) is superior to the complementary energy orthogonalization criterion (Eq. (13)).

## 8 Conclusion

Some quarter century ago, Roberts [23] noted that: "Because linear systems are so much easier to analyze than nonlinear ones, a natural method of attacking nonlinear problems is to replace a given set of nonlinear equations by an equivalent set of linear ones; the difference between the sets of equations is minimized in some appropriate sense." Likewise, one of the three pioneers of the stochastic linearization technique, Kazakov [2, p. 51] mentioned over half a century ago that: "The sufficiently accurate approximation of essentially nonlinear characteristics is extremely difficult." Hence it appears very important to search for the *nonclassical* criteria that may lead to better approximations.

The effectiveness of the energy concepts in the nonlinear stochastic dynamics is not accidental. Indeed, the expression of the exact probability density contains the expression of the potential energy.

In this study, we investigated a system with nonlinear restoring force. Extension of energy concepts can be performed for the systems with nonlinear damping. Such approaches have been initiated in papers by Li et al. [24], and Liang and Feeny [25]. It appears interesting to combine the present approach with that developed by Casciati et al. [26]. The applicability of the energy criteria ought to be explored to complex mechanical, civil, and aerospace structures.

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